Core Courses-12

Semester-v

Paper: Mechanics- I

Lesson: Moments of inertia of some rigid bodies

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Contents:

Chapter : Moments of inertia

1. Introduction : Moments of inertia plays the same role in rotational kinetics as the mass plays role in linear kinetics. So for further study of mechanics we have to know the moment of inertia of the different rigid bodies.

2: Rigid body: A rigid body is a solid body in which mutual distance between of every pair of specified particles in it is unchanged with respect to time. So a rigid body does not expand or contract i.e, in which deformation is neglected. A perfect rigid body does not exist. A metal rod can be taken as an example of rigid body.

3: Moments of inertia of a body about a line: A rigid body can be considered as a collection of particles. If m be the mass of a single element of a rigid body and x is the distance of the particle from the line the $\sum \mathrm{mx}$ is said to the moment of inertia of the body about the line.

4: Moment of inertia of some rigid bodies:

4.1:

AB be rod of length of 2l and mass M find the moment of inertia of the rod about an axis through an extremity and perpendicular to it ana about an axis through the middle point perpendicular to the rod.

Solution:

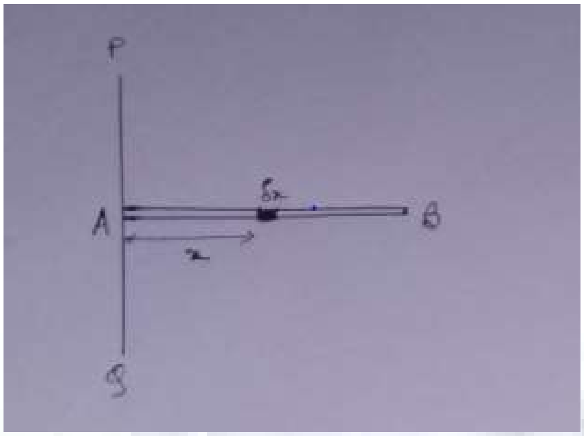
For the first part

M be the mass of the rod and ρ be the density per unit length

Then M= 2lρ

Let us take an element of breadth δx at a distance x from the end A

Mass of the element = $\rho\delta x$



Moment of inertia of the elementary mass about AL is =ρδxx^2

Hence the moment of inertia of the rod about AL is

$\int_0^{2l} \rho x^2 dx$ =ρ$\int_0^{2l} x^2 dx$

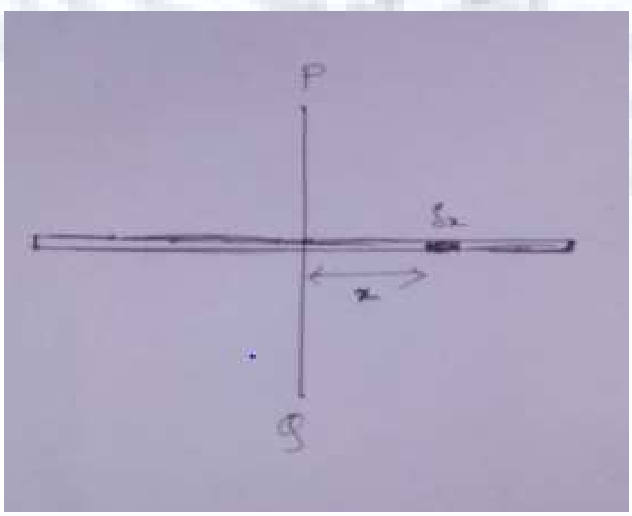
=ρ $[\frac{x^3}{3}]_0^{2l}$

=$\frac{\rho}{3} 8l^3$

=$\frac{M}{2a}\frac{8l^3}{3}$ [since M=2aρ]

=$\frac{4}{3}Ml^2$

For the second part



Moment of inertia of the element about LN is =ρδxx^2

Therefore the moment of inertia of the rod about LN is

$$= \int_0^{2l} \rho x^2 dx = \rho\ \rho\ [\frac{x^3}{3}]_{-l}^{l}$$

$$= \frac{\rho}{3} 2l^3$$

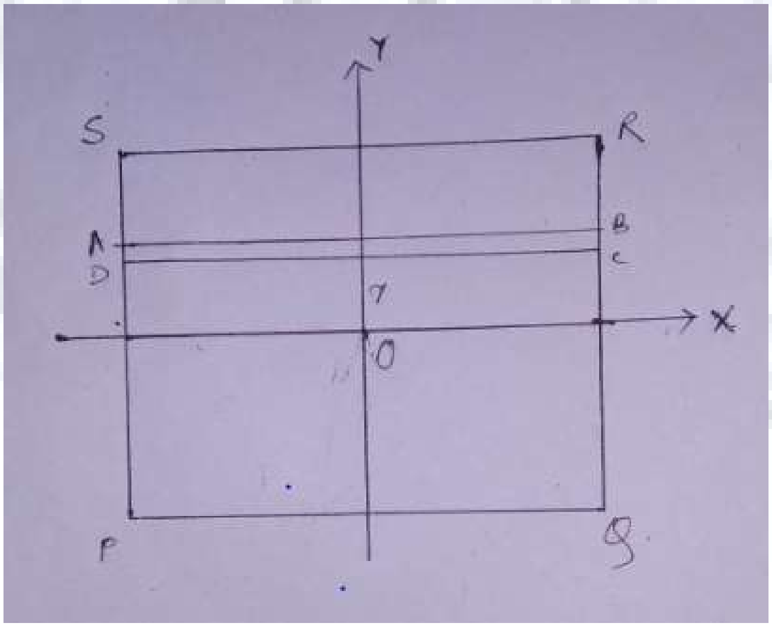
$$= \frac{M}{2a}\frac{2l^3}{3} \qquad \text{[since M=2aρ]}$$

$$= \frac{1}{3} M l^2$$

4.2

Find the moment of inertia of a rectangular lamina about a straight line through its centre and parallel to one of its edges.

Solution:



PQRS be a uniform rectangular lamina

Let PQ=2a, QR=2b and ρ be the surface density.

If M be the mass of the rod then M=2a.2b.ρ = 4abρ

O be the centre of the lamina and through O let OX and OY be axes of symmetry, which are parallel to PQ and QR, respectively.

To find the moment of inertia of the lamina at first we have consider an elementary strip ABCD of the lamina of breadth δy parallel to OX. Let y be the distance of the strip from OX.

So the mass of the strip is =2a δy ρ

Moment of inertia of the elementary strip about the OX = 2a δy ρy^2

Now we have to find the rectangular lamina about OX.

The moment of inertia of the lamina about OX is

$$=\int_{-b}^{b} 2a\rho y^2 dy$$

$$= 2a\rho\int_{-b}^{b} y^2 dy$$

$$=2a\rho[y^3/3]_{-b}^{b}$$

$$=2a\rho\frac{b^3+b^3}{3}$$

$$=\frac{4ab^3}{3}\rho$$

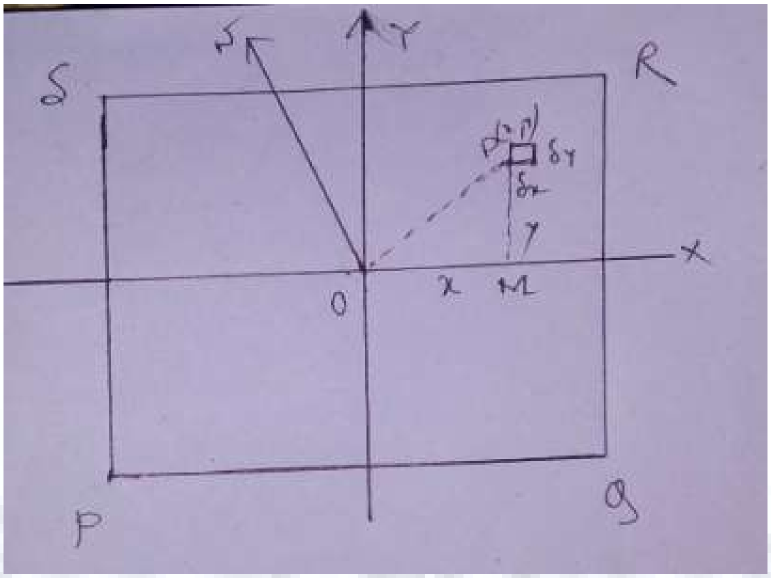
$$= \frac{4ab^3}{3}\frac{M}{4ab} \qquad \text{[since } M=4ab\rho]$$

$$=\frac{Mb^2}{3}$$

[if it is given that to find the moment of inertia of the above rectangular lamina about a line through its centre perpendicular to the plane the we have to consider an elementary area $\delta x\ \delta y$ at some point P(x,y).

The distance of the elementary area from O is $\sqrt{x^2 + y^2}$

Mass of the elementary area is $\rho\ \delta x\ \delta y$ and the moment of inertia of the element about the line ON is = $\rho\ \delta x\ \delta y(x^2 + y^2)$



Hence the moment of inertia be then

$$\rho\int_{-b}^{b}\int_{-a}^{a}(x^2+y^2)\mathrm{dxdy}$$

$$=\rho\int_{-a}^{a}[ax^2y+\frac{y^3}{3}]_{-b}^{b}\mathrm{dx}$$

$$=\rho\int_{-a}^{a}[2bx^2+\frac{2b^3}{3}]\mathrm{dx}$$

$$=\rho[\frac{2bx^3}{3}+\frac{2b^3x}{3}]_{-a}^{a}$$

$$=\rho[\frac{2b2a^3}{3}+\frac{2b^3 2a}{3}]$$

$$=\rho\frac{4ab}{3}(a^2+b^2)$$

$$=\frac{M}{4ab}\frac{4ab}{3}(a^2+b^2) \qquad \text{[since } M=4ab\rho]$$

$$=\frac{M}{3}(a^2+b^2)$$

4.3:]

Find the moment of inertia of a circular ring about (i) a diameter and (i) an axis through the centre of the ring perpendicular to its plane.

Solution:

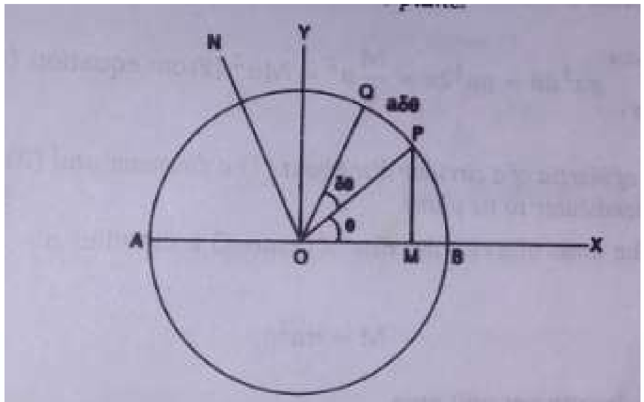


FIG 1.7

Reference: Rigid dynamics by Dr Md Motiur Rahman

let o be the centre of the ring and let OX be the initial line, AB be a diameter of the circular ring and a be the radius of the ring.

Let ρ be the density per unit length, then the mass of the ring is M=2πaρ (1)

Let P(a ,Ө) be point on the ring and let us consider an elementary arc of length a δӨ at P

Mass of the element is = ρ a δӨ

Distance of the element from the diameter AB is = $a\sin\Theta$.

So the moment of inertia of the element about AB is = a δӨ $(a\sin\Theta)^2$

Hence the moment of inertia of the ring about the diameter AB is 1

$$\int_0^{2\pi} \rho a\,(\sin\Theta)^2 d\Theta$$

$$=\rho a^3 \int_0^{2\pi} (\sin\Theta)^2\, d\Theta$$

$$=\frac{\rho a^3}{2}\int_0^{2a}(1-\cos 2\Theta)d\Theta$$

$$=\frac{\rho a^3}{2}[\,\Theta\ -\frac{\sin 2\Theta}{2}]_0^{2\pi}$$

$$=\frac{\rho a^3}{2}[2\pi]$$

$$=\pi\rho a^3$$

$$=\pi\frac{M}{2\pi a}a^3 \qquad \text{using (1)}$$

$$=\frac{1}{2}Ma^2$$

For the second part

Let ON be a line through the center of the ring perpendicular to the plane

Moment of inertia of the element at P about ON is = $a^2 \rho a \delta \Theta$

$= a^3 \rho \delta \Theta$

Therefore moment of inertia of the circular ring about ON is

$$= \int_0^{2\pi} a^3 \rho d\Theta$$

$$= a^3 \rho [\Theta]_0^{2\pi}$$

$$= a^3 \rho \, 2\pi$$

$$= a^3 \, 2\pi \frac{M}{2\pi a} \qquad \text{using (1)}$$

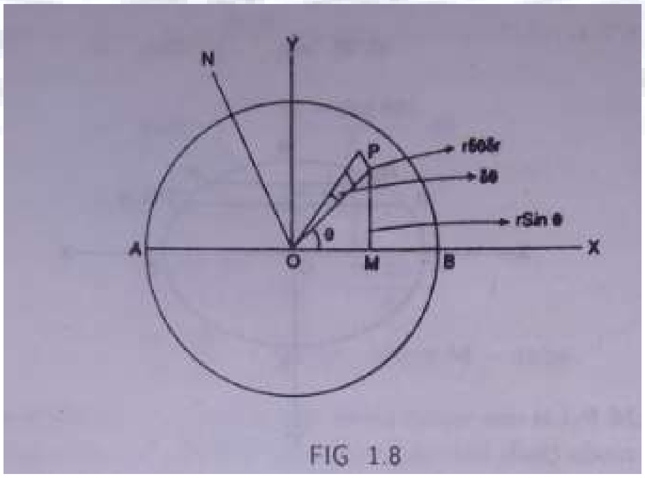
$$= Ma^2$$

Hence the moment of inertia of circular ring of radius a and mass M about (1) a diameter (2) a line through the centre perpendicular to the plane of a ring are $\frac{1}{2}Ma^2$ and Ma^2 respectively.

4.4:

Find the moment of inertia of a circular disc about (i) a diameter and (ii) and an axis through the centre perpendicular to its plane.

Solution:



Reference: Rigid dynamics by Dr Md Motiur Rahman

Let O be the centre of the disk and a be the radius of the disk . Let ρ be the density per unit area of the disk the the mass of the disk is M =$\pi a^2 \rho$

Let us consider an elementary area at P(r,Θ) of the disk with respect to O as ole and OX as initial line . area of the elementary area is rδΘδr

Mass of the elementary area is ρrδΘδr

Distance of this elementary area from OX is $\mathrm{r}\sin\Theta$ and the moment of inertia of the element is = $\rho r \delta\Theta \delta r (\mathrm{r}\sin\Theta)^2$

Hence the moment of inertia of the disk about the the line OX is

$$\int_0^{2\pi}\int_0^{a} \rho r^3\, sin^2\Theta d\Theta dr$$

$$=\rho\int_0^{2\pi}\int_0^{a} r^3 dr\, sin^2\Theta d\Theta$$

$$=\frac{\rho a^4}{4}\frac{1}{2}\int_0^{2\pi}(1-\cos 2\Theta)$$

$$=\frac{\rho a^4}{8}[\Theta-\frac{\sin 2\Theta}{2}]_0^{2\pi}$$

$$=\frac{\rho a^4}{8}2\pi$$

$$=\frac{M}{\pi a^2}\frac{\pi a^4}{4}$$

$$=\frac{1}{4}Ma^2$$

For the second part

Draw a line perpendicular to the plane of the disc through the centre of the of the disk O .

Distance of the element at P from the centre of the disk is r

Hence the moment of inertia of the elementary area about the line ON is

$$= \rho r \delta\Theta \delta r\, r^2$$

Therefore the moment of inertia of the circular disc about the line ON is

$$\int_0^{2\pi}\int_0^{a} \rho r^3\, d\Theta dr$$

$=\rho\int_0^{2\pi} d\Theta \int_0^a r^3\,dr$

$=2\pi\frac{a^4}{4}\rho$

$=\frac{\pi a^4}{2}\frac{M}{\pi a^2}$ since $M=\pi a^2\rho$

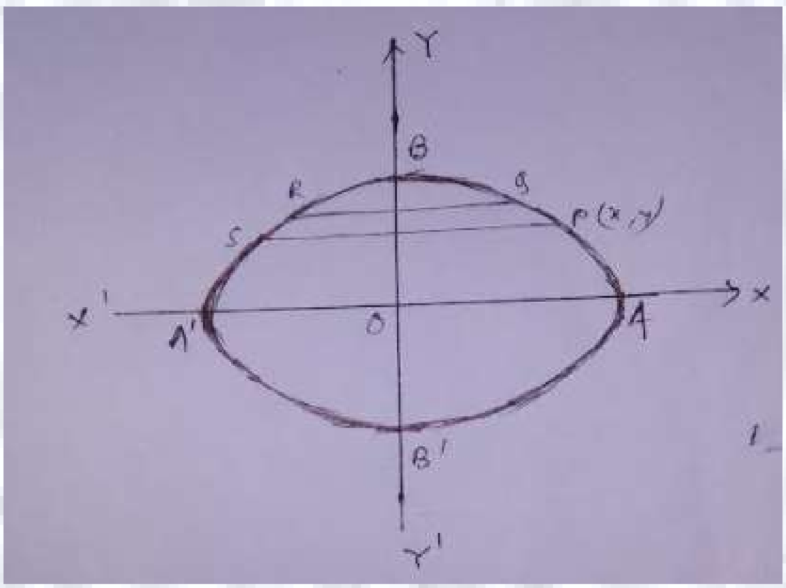
$=\frac{1}{2}Ma^2$

4.5:

Find the moment the of inertia of an elliptic disc about a major axis.

Solution

Let the equation of the ellipse be $\frac{x^2}{c^2}+\frac{y^2}{d^2}=1$ (1)



Let p(x,y) be a point on the ellipse and let us consider a elementary strip PQRS such that the distance of the centre is y and breadth δy

So the area of the strip is = $2x\delta y$

Mass of the elliptic plate is M= πabρ, where ρ is the density per unit area.

So the area of the strip is = $2x\delta y$,where x can be calculated from (1)

From (1)

$$\frac{x^2}{c^2}=1-\frac{y^2}{d^2}$$

$$\frac{x^2}{c^2} = \frac{d^2-y^2}{d^2}$$

$$x^2 = \frac{c^2}{d^2}(d^2 - y^2)$$

$$x = \frac{c}{d}\sqrt{(d^2 - y^2)}$$

So the area of the elementary strip can be written as $= 2\frac{c}{d}\sqrt{(d^2 - y^2)}\delta y$ and the mass be $= 2\rho\frac{c}{d}\sqrt{(d^2 - y^2)}\delta y$

Hence the moment of inertia of the strip about the major axis be = $2\rho\frac{c}{d}\sqrt{(d^2 - y^2)}\delta y\, y^2$

Therefore the moment of inertia of the elliptic disc about the major axis is

$$= \int_{-d}^{d} 2\rho\frac{c}{d}\sqrt{(d^2 - y^2)}\, y^2 dy$$

$$= \frac{2c}{d}\int_{-d}^{d}\sqrt{(d^2 - y^2)}y^2 dy \qquad (2\}$$

Let $y = d\sin\Theta$

y	Θ
-d	$-\frac{\pi}{2}$
d	$\frac{\pi}{2}$

The (2) becomes

$$= \frac{2c\rho}{d}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d^4\sqrt{1 - sin^2\Theta}\, sin^2\Theta cos\Theta d\Theta$$

$$= 2c\rho d^3\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} sin^2\Theta cos^2\Theta d\Theta$$

$$= \rho c d^3\int_{0}^{\frac{\pi}{2}} 4\, sin^2\Theta cos^2\Theta d\Theta$$

$$= \rho c d^3\int_{0}^{\frac{\pi}{2}} sin^2 2\Theta d\Theta$$

$$= \frac{\rho c d^3}{2}\int_{0}^{\frac{\pi}{2}}(1 - cos2\Theta)\, d\Theta$$

$$= \frac{\rho c d^3}{2}[\Theta - \frac{sin2\Theta}{2}]_0^{\frac{\pi}{2}}$$

$$= \frac{\rho c d^3}{2}\frac{\pi}{2}$$

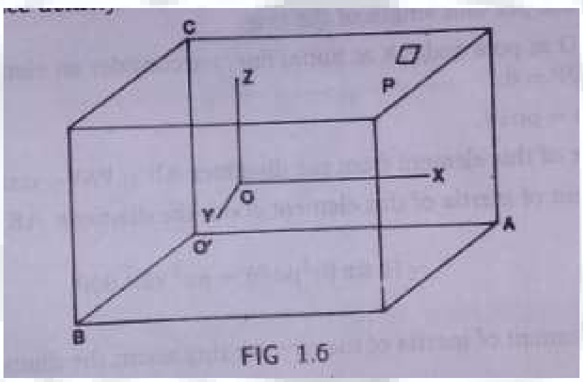
$$= \frac{\pi c^{\ 3}}{4}\frac{M}{\pi c d}$$ since M =πcd

$$= \frac{M d^2}{4}$$

4.6:

Find the moment of inertia of a rectangular parallelopiped about an axis through its centre and parallel to one of the sides.

Solution:



Reference: Rigid dynamics by Dr Md Motiur Rahman

Let 2p, 2q ,2r are the lengths of the sides of the parallelopiped. Also let through the centre O of the parallelopiped OX, OY, OZ axes has drawn parallel to ages of length 2p,2q, 2r

Let ρ be the suface density of and therefore the mass of the paralleopiped is M=2p2q2rρ

Now we consider an elementary volume δxδyδz at P(x,y,z),then the mass of the volume is δm=ρδxδyδz.

We are trying to calculate the moment of inertia of the parallelopiped about the axis which is parallel to side of length 2p.

The distance of the element from the axis OX is $\sqrt{(y^2+z^2)}$ and hence the moment of inertia of the element about OX is = δm(x^2+y^2)

Therefore the moment of inertia of the parallelopiped about OX is

$$= \int_{-p}^{p}\int_{-q}^{q}\int_{-r}^{r} \rho(y^2+z^2)dxdydz$$

$$= \rho\int_{-q}^{q}\int_{-r}^{r} 2p(y^2+z^2)\,dydz$$

$$= \rho 2p\int_{-r}^{r}[\frac{y^3}{3}+z^2y]_{-q}^{q}dz$$

$$= 2p\rho\int_{-r}^{r}(\frac{2q^3}{3}+2qz^2)dz$$

$$= 2p\rho[\frac{2q^3z}{3}+\frac{2qz^3}{3}]_{-r}^{r}$$

$$= \frac{4p\rho}{3}[2q^3r+2qr^3]$$

$$= \frac{8pqr\rho}{3}(q^2+r^2)$$

$$= \frac{M}{3}(q^2+r^2)$$

Hence the moment of inertia of rectangular parallelopiped about an axis through its center and parallel to one of its edges is

$$\frac{total\ mass}{3}(sum\ of\ the\ squares\ of\ half\ the\ length\ of\ the\ non\ parallel\ edges\ with\ the\ axis\ )$$

6:References : **(1)** Dr Motiur Rahman, Rigid Dynamics .

(2) S.L. Loney , Dynamics of a particle and of rigid bodies.